THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A/B (First Term, 2020-21)
Linear Algebra II
Solution to Homework 10

Sec. 6.3

Q3(b). For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

$$\mathrm{V} = \mathrm{C}^2,\ \mathrm{T}\left(z_1, z_2\right) = \left(2z_1 + iz_2, (1-i)z_1\right), x = (3-i, 1+2i)$$

Sol. Solution: Denote $\beta = \{(1,0),(0,1)\}$ as the standard ordered basis for V under field $F = \mathbb{C}$. Then

$$[T]_\beta = \left(\ [T(1,0)]_\beta \quad [T(0,1)]_\beta\ \right) = \begin{pmatrix} 2 & i \\ 1-i & 0 \end{pmatrix} \quad \Rightarrow \quad [T^*]_\beta = ([T]_\beta)^* = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}$$

It follows that

$$[T^*(x)]_\beta = [T^*]_\beta\,[x]_\beta = \begin{pmatrix} 2 & 1+i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 3-i \\ 1+2i \end{pmatrix} = \begin{pmatrix} 5+i \\ -1-3i \end{pmatrix}$$

Therefore, $T^*(x) = (5+i, -1-3i)$

Q3(c). $V = P_1(\mathbb{R})$ with $\langle f, g\rangle = \displaystyle\int_{-1}^{1} f(t)g(t)dt$, $T(f) = f' + 3f$. $f(t) = 4-2t$.

Sol. Let $\beta = \{1, t\}$ be the standard basis of V. Write $T^*(4-2t) = a + bt$ for some $a, b \in \mathbb{R}$. Then for any $g(t) = c + dt \in V$ with $c, d \in \mathbb{R}$, we have $T(g(t)) = d + 3c + 3dt$ and

$$\langle d + 3c + 3dt, 4-2t\rangle = \langle T(g(t), 4-2t\rangle = \langle g(t), T^*(4-2t)\rangle = \langle c + dt, a + bt\rangle\,.$$

Now $\langle d+3c+3dt, 4-2t\rangle = 2(4)(d+3c) + (3d)(-2)\frac{2}{3} = 4d + 24c$ and $\langle c+dt, a+bt\rangle = 2ac + \frac{2}{3}bd$. Since c, d are arbitrary, the coefficients of them on both sides of the equation must equal respectively. Therefore $24 = 2a$ and $\frac{2}{3}b = 4$. Hence $a = 12$ and $b = 6$. So $T^*(4-2t) = 12 + 6t$.

Q6. Let T be a linear operator on an inner product space V. Let $U_1 = T + T^*$ and $U_2 = TT^*$. Prove that $U_1 = U_1^*$ and $U_2 = U_2^*$.

Sol.

$$U_1^* = (T + T^*)^* = T^* + (T^*)^* = T^* + T = U_1.$$
$$U_2^* = (TT^*)^* = (T^*)^*T^* = TT^* = U_2.$$

Q8. Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Prove that if T is invertible, then T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

Sol. Suppose $x \in \mathsf{N}(T^*)$. Then

$$0 = \left\langle T^{-1}(x), T^*(x) \right\rangle = \left\langle TT^{-1}(x), x \right\rangle = \langle x, x \rangle .$$

Hence $x = \overrightarrow{0}$ and thus T^* is an injective linear operator on V. So T^* is invertible by finiteness of dimension of V. Also we have

$$\left\langle x, (T^{-1})^*(y) \right\rangle = \left\langle T^{-1}(x), T^*(T^*)^{-1}(y) \right\rangle = \left\langle TT^{-1}(x), (T^*)^{-1}(y) \right\rangle = \left\langle x, (T^*)^{-1}(y) \right\rangle$$

for all $x, y \in V$. Therefore $(T^*)^{-1} = (T^{-1})^*$.

Q9. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$. Hint: Recall that $\mathsf{N}(T) = W^\perp$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

Sol. From the assumption $V = W \oplus W^\perp$, for all $v, w \in V$, there exist unique $v_1, w_1 \in W$ and $v_2, w_2 \in W^\perp$ such that $v = v_1 + v_2$ and $w = w_1 + w_2$. We check that

$$\langle T(v), w \rangle = \langle v_1, w_1 + w_2 \rangle = \langle v_1, w_1 \rangle + \langle v_1, w_2 \rangle = \langle v_1, w_1 \rangle$$

and so

$$\langle v, T(w) \rangle = \overline{\langle T(w), v \rangle} = \overline{\langle w_1, v_1 \rangle} = \langle v_1, w_1 \rangle = \langle T(v), w \rangle .$$

Therefore T^* exists and $T = T^*$.

Q13. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following results.

(a) $\mathsf{N}(T^*T) = \mathsf{N}(T)$. Deduce that $\operatorname{rank}(T^*T) = \operatorname{rank}(T)$.

(b) $\operatorname{rank}(T) = \operatorname{rank}(T^*)$. Deduce from (a) that $\operatorname{rank}(TT^*) = \operatorname{rank}(T)$.

(c) For any $n \times n$ matrix A. $\operatorname{rank}(A^*A) = \operatorname{rank}(AA^*) = \operatorname{rank}(A)$.

Sol. (a) It is clear that $\mathsf{N}(T) \subset \mathsf{N}(T^*T)$. Let $x \in \mathsf{N}(T^*T)$. Then $\langle T(x), T(x) \rangle = \langle x, T^*T(x) \rangle = \langle x, \overrightarrow{0} \rangle = 0$. Hence $T(x) = \overrightarrow{0}$ and $x \in \mathsf{N}(T)$. It follows that

$$\operatorname{rank}(T^*T) = n - \operatorname{nullity}(T^*T) = n - \operatorname{nullity}(T) = \operatorname{rank}(T)$$

where $n = \dim(V)$.

(b) By Q12(b), $\mathsf{R}(T^*) = \mathsf{N}(T)^\perp$. Since $V = \mathsf{N}(T) \oplus \mathsf{N}(T)^\perp$ by Sec 6.2 Q13(d), we have $n = \operatorname{nullity}(T) + \dim(\mathsf{N}(T)^\perp)$ and

$$\operatorname{rank}(T^*) = \dim(\mathsf{N}(T)^\perp) = n - \operatorname{nullity}(T) = \operatorname{rank}(T).$$

(c) Note that $L_A^* = L_{A^*}$. Hence by applying part (a) and (b) with $T = L_A$, we have $\operatorname{rank}(A^*A) = \operatorname{rank}(L_{A^*}L_A) = \operatorname{rank}(L_A^*L_A) = \operatorname{rank}(L_A) = \operatorname{rank}(A)$. Similarly, $\operatorname{rank}(AA^*) = \operatorname{rank}(A)$.

Q14. Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

Sol. For all $x, w \in V$, we have

$$\langle T(x), w\rangle = \langle\langle x, y\rangle z, w\rangle = \langle x, y\rangle \langle z, w\rangle = \left\langle x, \overline{\langle z, w\rangle} y\right\rangle = \langle x, \langle w, z\rangle y\rangle .$$

Note that $w \mapsto \langle w, z\rangle y$ is a linear operator on V since

$$\langle w_1 + cw_2, z\rangle y = (\langle w_1, z\rangle + c\langle w_2, z\rangle)y = \langle w_1, z\rangle y + c\langle w_2, z\rangle y$$

for all w_1, $w_2 \in V$ and scalar c. Therefore this gives the adjoint of T.

Sec. 6.4

2. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c). $V = \mathbb{C}^2$ and T is defined by $T(a, b) = (2a + ib, a + 2b)$

Sol. Take $\beta = \{(1, 0), (0, 1)\}$ as the ordered basis for V. Then

$$[T]_\beta = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \quad \Rightarrow \quad [T^*]_\beta = ([T]_\beta)^* = \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix}$$

Therefore, we have

$$[T^*T]_\beta = [T^*]_\beta [T]_\beta = \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2+2i \\ 2-2i & 5 \end{pmatrix}$$

and also

$$[TT^*]_\beta = [T]_\beta [T^*]_\beta = \begin{pmatrix} 2 & i \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -i & 2 \end{pmatrix} = \begin{pmatrix} 5 & 2+2i \\ 2-2i & 5 \end{pmatrix} = [T^*T]_\beta$$

As the matrix representation map is an isomorphism, we have $T^*T = TT^*$, i.e. T is normal. Also, as $[T]_\beta \neq [T^*]_\beta$, $T \neq T^*$ and hence T is not self-adjoint operator. We then solve for the eigenvalue of T Consider $f_T(t) = \det([T]_\beta - \lambda I_2) = \det \begin{pmatrix} 2-\lambda & i \\ 1 & 2-\lambda \end{pmatrix} = (\lambda-2)^2 - i = 0$. Solving $\lambda - 2 = \sqrt{i} = \pm\frac{\sqrt{2}}{2}(1+i)$. Then we have the eigenvalue given by $\lambda_1 = 2 + \frac{\sqrt{2}}{2}(1+i)$ and $\lambda_2 = 2 - \frac{\sqrt{2}}{2}(1+i)$

For $\lambda_1 = 2 + \frac{\sqrt{2}}{2}(1+i)$, consider

$$E_{\lambda_1} = N(T - \lambda_1 I_2) = N\begin{pmatrix} 2 - \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & 2 - \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} = \left\{ t \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2}(1-i) \end{pmatrix} : t \in \mathbb{C} \right\}$$

Obviously we have $\left\| \left(\frac{\sqrt{2}}{2}, \frac{1}{2}(1-i) \right) \right\| = 1$

For $\lambda_2 = 2 - \frac{\sqrt{2}}{2}(1+i)$. consider

$$E_{\lambda_2} = N\left(T - \lambda_2 I_2\right) = N\begin{pmatrix} 2 + \frac{\sqrt{2}}{2}(1+i) & i \\ 1 & 2 + \frac{\sqrt{2}}{2}(1+i) \end{pmatrix} = \left\{ t \begin{pmatrix} -\frac{1}{2}(1+i) \\ \frac{\sqrt{2}}{2} \end{pmatrix} : t \in \mathbb{C} \right\}$$

Obviously we have $\left\| \left(-\frac{1}{2}(1+i), \frac{\sqrt{2}}{2} \right) \right\| = 1$ Therefore, we can take the orthonormal basis of eigenvectors of T for V can be taken as

$$\left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{1}{2}(1-i) \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}(1+i) \\ \frac{\sqrt{2}}{2} \end{pmatrix} \right\}$$

(d). $V = P_2(\mathbb{R})$ and T is defined by $T(f) = f'$, where

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

Sol. Solution: Take $\alpha = \left\{1, x, x^2\right\}$ as the orthogonal basis for V and hence we can apply G-S process to obtain the ordered orthonormal basis for $V, \beta = \left\{1, 2\sqrt{3}(t - 1/2), 6\sqrt{5}\left(t^2 - t + 1/6\right)\right\}$ Check by definition we can obtain

$$[T]_\beta = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad [T^*]_\beta = ([T_\beta])^* = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix} \neq [T]_\beta$$

Therefore, T is not an adjoint linear operator. Also, we have

$$[TT^*]_\beta = [T]_\beta [T^*]_\beta = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 60 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$[T^*T]_\beta = [T^*]_\beta [T]_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix} \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 60 \end{pmatrix} \neq [T^*T]_\beta$$

Therefore, $T^*T \neq TT^*$ and hence T is not normal operator. So, there exist no orthonormal basis of eigenvectors of T for V.

6. Q: Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}\left(T + T^*\right) \quad \text{and} \quad T_2 = \frac{1}{2i}\left(T - T^*\right)$$

(a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.

(b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$

(c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

Sol: (a) We have

$$T_1^* = \left(\frac{1}{2}(T + T^*)\right)^* = \left(\frac{1}{2}\right)^* (T^* + (T^*)^*) = \frac{1}{2}(T^* + T) = \frac{1}{2}(T + T^*) = T_1$$

T_1 is self-adjoint. Also, we have

$$\begin{aligned} T_2^* &= \left(\frac{1}{2i}(T - T^*)\right)^* = \left(-\frac{i}{2}(T - T^*)\right)^* = \frac{i}{2}(T - T^*)^* \\ &= \frac{i}{2}(T^* - T) = \frac{i^2}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = T_2 \end{aligned}$$

so T_2 is also self-adjoint. It is clear that

$$T_1 + iT_2 = \frac{1}{2}(T + T^*) + i\left[\frac{1}{2i}(T - T^*)\right] = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) = T$$

(b) From assumption, we have $T = T_1 + iT_2 = U_1 + iU_2$ and hence

$$(T_1 - U_1) + i(T_2 - U_2) = 0 \tag{1}$$

As T_1, T_2, U_1, U_2 are self-adjoint, takeing adjoint operator on both sides

$$(T_1 - U_1) - i(T_2 - U_2) = (T_1^* - U_1^*) - i(T_2^* - U_2^*) = ((T_1 - U_1) + i(T_2 - U_2))^* = 0 \tag{2}$$

Adding (1) and (2) to use

$$2(T_1 - U_1) = 0 \Rightarrow T_1 = U_1$$

Consider $(1) - (2) : 2i(T_2 - U_2) = 0$ yields $T_2 = U_2$. The proof is completed.

(c) ($\Rightarrow$) Suppose T is normal, then

$$\begin{aligned} T_1^2 + iT_1T_2 - iT_2T_2 + T_2^2 &= (T_1 - iT_2)(T_1 + iT_2) = (T_1 + iT_2)^*(T_1 + iT_2) \\ &= T^*T = TT^* = (T_1 + iT_2)(T_1 + iT_2)^* = (T_1 + iT_2)(T_1 - iT_2) \\ &= T_1^2 - iT_1T_2 + iT_2T_1 + T_2^2 \end{aligned}$$

By swapping the terms in the equality above yields $2iT_1T_2 = 2iT_2T_1$ and hence $T_1T_2 = T_2T_1$.

($\Leftarrow$) Suppose $T_1T_2 = T_2T_1$, we have

$$\begin{aligned} T^*T &= (T_1 + iT_2)^*(T_1 + iT_2) = (T_1 - iT_2)(T_1 + iT_2) = T_1^2 + T_2^2 + iT_1T_2 - iT_2T_1 \\ &= T_1^2 + T_2^2 + iT_2T_1 - iT_1T_2 = (T_1 + iT_2)(T_1 - iT_2) = (T_1 + iT_2)(T_1 + iT_2)^* = TT^* \end{aligned}$$

Hence T is normal operator.

7. Q: Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

(a) If T is self-adjoint, then T_W is self-adjoint.

(b) $W^\perp$ is T^*-invariant.

(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.

(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

Sol: (a) $\forall u, v \in W$, since T is self-adjoint,

$$\langle T_W(u), v\rangle = \langle T(u), v\rangle = \langle u, T(v)\rangle = \langle u, T_W(v)\rangle,$$

whence T_W is self-adjoint.

(b) Fix $w' \in W^\perp$ and $w \in W$. As W is T-invariant, $T(w) \in W$. Then

$$\langle w, T^*(w')\rangle = \langle T(w), w'\rangle = 0.$$

Therefore, $T^*(w) \in W^\perp$. $W^\perp$ is T^*-invariant.

(c) Fix $w \in W$. We claim that $(T_W)^*(w) = (T^*)_W(w)$. If suffices to show that $\forall w' \in W$, $\langle w', (T_W)^*(w)\rangle = \langle w', (T^*)_W(w)\rangle$. Indeed, $\forall w' \in W$,

$$\langle w', (T_W)^*(w)\rangle = \langle T_W(w'), w\rangle = \langle T(w'), w\rangle = \langle w', T^*(w)\rangle = \langle w', (T^*)_W(w)\rangle.$$

Therefore, $(T_W)^* = (T^*)_W$.

(d) We have $T_W(T_W)^* = T_W(T^*)_W = (TT^*)_W = (T^*T)_W = (T^*)_W T_W = (T_W)^* T_W$. Therefore, T_W is normal.

9. Q: Let T be a normal operator on a finite-dimensional inner product space V. Prove that $\mathsf{N}(T) = \mathsf{N}(T^*)$ and $\mathsf{R}(T) = \mathsf{R}(T^*)$.

Sol: Fix $v \in \mathsf{N}(T)$. If $v = \vec{0}$, then clearly $v \in \mathsf{N}(T^*)$. If $v \neq \vec{0}$, then v is an eigenvector of T corresponding to eigenvalue 0 and by Theorem 6.15, v is also an eigenvector of T^* corresponding to eigenvalue $\overline{0} = 0$, implying that $v \in \mathsf{N}(T^*)$. We have $\mathsf{N}(T) \subset \mathsf{N}(T^*)$. Note that T^* is also normal. Applying the above argument on T^* yields $\mathsf{N}(T^*) \subset \mathsf{N}((T^*)^*) = \mathsf{N}(T)$. Hence, $\mathsf{N}(T) = \mathsf{N}(T^*)$.
By Exercise 12 in Sec. 6.3, $\mathsf{R}(T^*) = \mathsf{N}(T)^\perp = \mathsf{N}(T^*)^\perp = \mathsf{R}((T^*)^*) = \mathsf{R}(T)$.

11. Q: Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

(a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.

(b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$. Hint: Replace x by $x + y$ and then by $x + iy$ and expand the resulting inner products.

(c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$

Sol: (a) As T is self-adjoint, i.e.

$$\langle T(x), x\rangle = \langle x, T^*(x)\rangle = \overline{\langle T^*(x), x\rangle} = \overline{\langle T(x), x\rangle}$$

Therefore, we have

$$\langle T(x), x\rangle = \frac{1}{2}(\langle T(x), x\rangle + \overline{\langle T(x), x\rangle}) = \frac{1}{2} \cdot 2\,\mathrm{Re}(\langle T(x), x\rangle) = \mathrm{Re}(\langle T(x), x\rangle) \in \mathbb{R}$$

The proof is completed.

(b) Pick $x, y \in V$, we have $\langle T(x), x\rangle = 0$ and $\langle T(y), y\rangle = 0$. Also, as $x+y \in V$, it follows that

$$0_v = \langle T(x+y), x+y\rangle = \langle T(x)+T(y), x+y\rangle = \langle T(x), y\rangle + \langle T(y), x\rangle \tag{3}$$

Similarly, as $x+iy \in V$, we have

$$0 = \langle T(x+iy), x+iy\rangle = \langle T(x)+iT(y), x+iy\rangle = \bar{i}\langle T(x), y\rangle + i\langle T(y), x\rangle = -i\langle T(x), y\rangle + i\langle T(y), x\rangle \tag{4}$$

And hence (5): $0 = \langle T(x), y\rangle - \langle T(y), x\rangle$. Summing (3) and (5) yields $2\langle T(x), y\rangle = 0$ and so $\langle T(x), y\rangle = 0$. As this statement holds for all $x, y \in V$, we have $T = T_0$.

(c) Suppose $\langle T(x), x\rangle \in \mathbb{R}$ for all $x \in V$

$$\langle T(x), x\rangle = \langle x, T^*(x)\rangle = \overline{\langle T^*(x), x\rangle} \overset{(\star)}{=} \langle T^*(x), x\rangle \quad \Rightarrow \quad \langle (T - T^*)(x), x\rangle = 0$$

where $(\star)$ holds because taking conjugation on real number does not change the value. As $\langle (T - T^*)(x), x\rangle = 0$ for all $x \in V$, it follows by (b) that $T - T^* = T_0$. Therefore, we have $T = T^*$